Nonconvex Game Theory

DEI Doctoral Research Seminars - February 2nd 2022

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Presentation Outline

1. Introduction to Convex Optimization and Game Theory
2. Monotone Operators and Variational Inequality Theory
3. Game Theory on Nonconvex Sets
4. Motivating Examples and Analysis of Case Studies
5. Conclusions and Research Outlooks

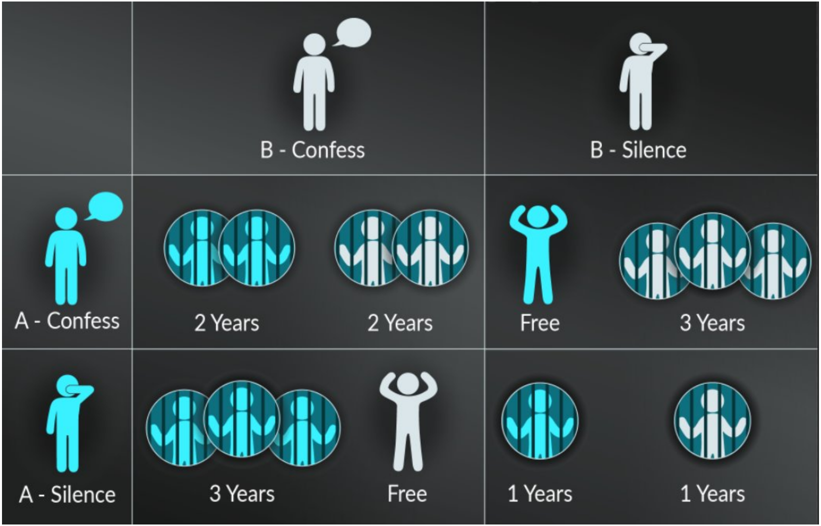
Introduction to Convex Optimization and Game Theory

Motivating example: the prisoner dilemma

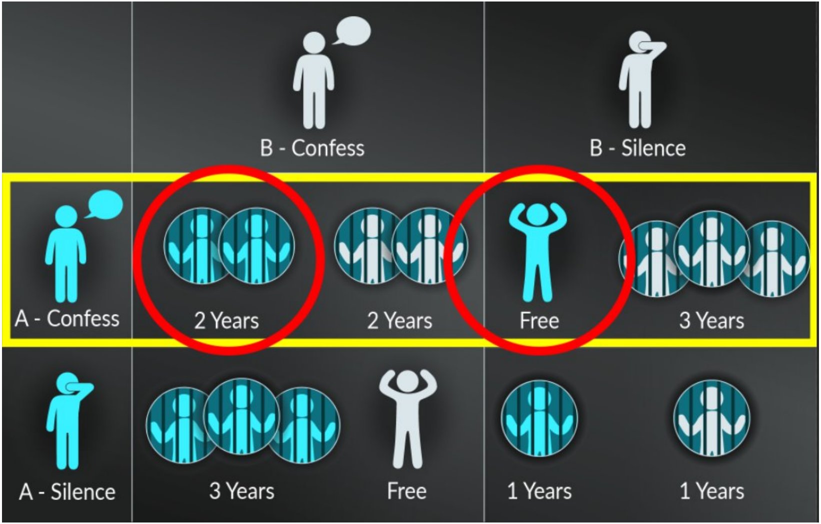
The police arrests two suspect and put them in separate rooms so that they cannot talk to each other (cannot coordinate).

- **If both confess:** each gets 2 years in jail
- **If neither confess:** each gets 1 year in jail
- **If one confesses, while the other does not:** the first is free while the other gets 3 years in jail

Motivating example: the prisoner dilemma



Motivating example: the prisoner dilemma



Motivating example: the prisoner dilemma

Can we mathematically model this situation?

YES!

How? $\Rightarrow$ Game Theory

Game Theory: a formal definition

There are some key aspects in prisoner dilemma example:

- **Players or decision-makers**: there is a set of N players who partecipate in the game, indexed by $i \in \mathcal{N} := \{1, ..., N\} \subseteq \mathbb{N}$
- **Actions or strategies**: each player can decide a "strategy", i.e., a decision variables $\mathbf{x}_i \in \mathbb{R}^n$
  - the strategies that each player can choose may be limited, i.e., exists a feasible set $\mathbf{x}_i \in \Omega_i \subseteq \mathbb{R}^n$
- **Outcomes or payoff functions**: players choose the strategy to achieve the best "outcome" that is dependent on the strategy, each player $i \in \mathcal{N}$ tries to minimize/maximize a payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbb{R}^n \times \mathbb{R}^{(N-1)n} \to \mathbb{R}$
  - the "outcome" is dependent also on $\mathbf{x}_{-i} := \text{col}(\mathbf{x}_1, ..., \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, ..., \mathbf{x}_N) \in \mathbb{R}^{(N-1)n}$, i.e., the strategies of the other players

Game Theory: a formal definition

Definition: Game theory (Vasile and Becerra 2014)

Game theory is a branch of mathematics that deals with situations where two or more individuals (called players) make decisions that affect each other.

Among the several types of games, we focus on games that are (1) noncooperative, (2) static, and with (3) perfect information.

Solution of a game: best response

In all game theory games, players choose strategies without knowing with certainty what the opposing player will do.

"I know what other opponent did but I don't know what they will do"

- ▷ Therefore, players act with BEST RESPONSES, the optimal strategy that maximizes a player's payoff given its beliefs about its rivals' strategies;
- ▷ A special kind of Best Response is a DOMINANT STRATEGY, a strategy that is a best response to all possible strategies that a rival might use.

Solution of a game: Nash equilibrium

The optimum solution for a game is the dominant strategy. However, this is not always available and thus, an alternative solution is an **equilibrium point**.

Theorem: Nash equilibrium (Nash 1950)

A set of strategies is a Nash equilibrium (NE) if no player can obtain a higher payoff by **unilaterally** choosing a different strategy.

A NE is self-enforcing: no player would want to deviate by choosing a different strategy *“given the strategies chosen by my rivals, I made the best possible choice”*

- a dominant strategy solution is a NE;
- at a NE the best responses line up;
- if multiple NE exist, we can’t conclude which outcome will occur;
- sometimes no NE exist.

Nash equilibrium problem

The *Nash equilibrium problem* (NEP) is thus defined with the inter-dependent optimization problems as follows:

$$\forall i \in \mathcal{N}: \quad \begin{cases} \min_{\mathbf{x}_i} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \Omega_i \end{cases}$$

Definition: Nash equilibrium (Dutang 2013)

A Nash equilibrium (NE) is a collective strategy profile $\mathbf{x}^* \in \Omega$ with the property that no single player $i \in \mathcal{N}$ can benefit from a unilateral deviation from $\mathbf{x}_i^*$ if all the other players act according to the NE. More formally:

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf \left\{ f_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \mid \mathbf{x}_i \in \Omega_i \right\}, \quad \forall i \in \mathcal{N}.$$

Generalized Nash equilibrium

- **Players or decision-makers**: there is a set of N players who partecipate in the game, indexed by $i \in \mathcal{N} := \{1, \dots, N\} \subseteq \mathbb{N}$
- **Actions or strategies**: each player can decide a "strategy", i.e., a decision variables $\mathbf{x}_i \in \mathbb{R}^n$
- **Outcomes or payoff functions**: players choose the strategy to achieve the best "outcome" that is dependent on the strategy, each player $i \in \mathcal{N}$ tries to minimize/maximize a payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbb{R}^n \times \mathbb{R}^{(N-1)n} \to \mathbb{R}$
- the strategies that each player can choose are limited by the strategies of the other players $\mathbf{x}_{-i} := \mathrm{col}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N) \in \mathbb{R}^{(N-1)n}$, i.e., exists a feasible set $\mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i})$

Generalized Nash equilibrium

The interaction between the players take place (also) at the feasible set level $\Rightarrow$ **shared feasible set**

$$\mathcal{X} = \Omega \cap \left\{ \mathrm{x} \in \mathbb{R}^{Nn} \mid g(\mathrm{x}) \leq \mathbf{0}_M \right\}$$

The Generalized Nash equilibrium (GNE) is the solution of:

$$\forall i \in \mathcal{N} : \quad \begin{cases} \min\limits_{\mathrm{x}_i} & f_i(\mathrm{x}_i, \mathrm{x}_{-i}) \\ \text{s.t.} & \mathrm{x}_i \in \mathcal{X}_i(\mathrm{x}_{-i}). \end{cases}$$

Definition: Generalized Nash equilibrium (Belgioioso et al. 2019)

A GNE is a collective strategy $\mathrm{x}^* \in \mathcal{X}$ such that for each $i \in \mathcal{N}$:

$$f_i(\mathrm{x}_i^*, \mathrm{x}_{-i}^*) \leq \inf \left\{ f_i(\mathrm{x}_i, \mathrm{x}_{-i}^*) \,|\, \mathrm{x}_i \in \mathcal{X}_i(\mathrm{x}_{-i}^*) \right\}, \forall i \in \mathcal{N}.$$

Existence of a generalized Nash equilibrium

How to prove the existence of a generalized Nash equilibrium?

...is quite difficult, we need some assumptions...

- **Continuity** of the payoff functions $f_i(\mathrm{x}_i, \mathrm{x}_{-i})$ and **compactness** of sets Ω_i assumptions are indispensable, usually needed for the existence of solutions to optimization problems.
- **Convexity** is usually required for fixed-point theorems that are used to demonstrate the existence of an equilibrium.

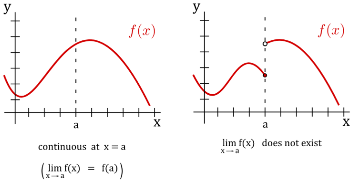
Let us introduce some formal prerequisites...

Continuity and Compactness

Definition: Continuity

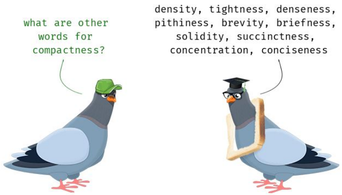
A function f is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a)$$



Definition: Compactness

A set $\Omega \subseteq \mathbb{R}^n$ is compact if every sequence in Ω has a subsequence that converges to a point in Ω.



simple version: in Euclidean spaces, a set is compact if it is **closed** and **bounded**

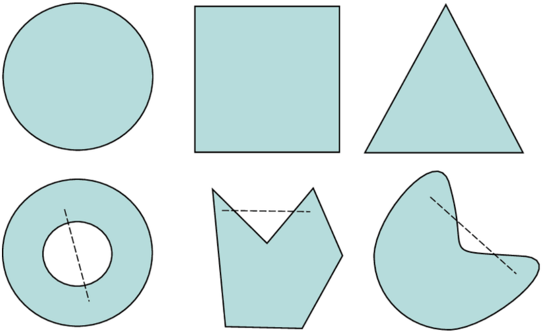
Convexity

Definition: Convex sets

A set $\mathcal{X}$ is convex if for any two points $x, y \in \mathcal{X}$, the segment joining them belongs to $\mathcal{X}$, i.e., $\alpha x + (1-\alpha)y \in \mathcal{X}$, $\forall x, y \in \mathcal{X}$ and $\alpha \in [0,1]$

Convex

Nonconvex



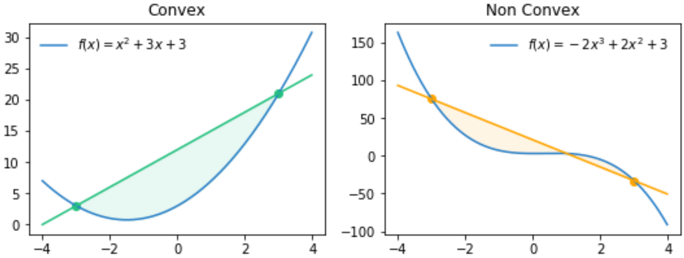
Convexity

Definition: Convex functions

Given a convex set $\mathcal{X}$, $\forall x, y \in \mathcal{X}$ and $\alpha \in (0, 1)$, a function $f(x)$ is:

- (strictly) convex on $\mathcal{X}$ if: $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$
- strongly convex if there exists $c > 0$ such that: $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \frac{c}{2}\alpha(1-\alpha)||x-y||^2$

Obviously: strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex



Existence of a generalized Nash equilibrium

convex
shared set
+
convex
payoff
functions

A GNE exists

Optimality conditions

From an optimization point of view: a point $\mathbf{x}_i^*$ is said to be an optimal solution for a player if:

$$f(\mathbf{x}_i^*) \leq f(\mathbf{x}_i), \quad \forall \mathbf{x}_i \in \mathcal{X}_i(\mathbf{x}_{-i}^*)$$

when a point is an optimal solution? $\Rightarrow$ **optimality conditions**

Definition: Minimum principle for single valued functions

A feasible point $\mathbf{x}_i^*$ is an optimal solution if and only if

$$(\mathbf{y} - \mathbf{x}_i^*)^\top \nabla f(\mathbf{x}_i^*) \geq 0, \quad \forall \mathbf{y} \in \mathcal{X}_i(\mathbf{x}_{-i}^*)$$

the minimum principle is equivalent to the Karush-Kuhn-Tucker (KKT) optimality conditions when the set is defined by inequalities and equalities

Monotone Operators and Variational Inequality Theory

Variational inequality problem

With the **variational inequality problem** we can ”generalize” the minimum principle replacing the gradient f with a general map G.

Definition: Variational inequality problem

Given a subset $K \subseteq \mathbb{R}^n$ and a map $G : K \to \mathbb{R}^n$, the *variational inequality problem* $\mathrm{VI}(K, G(\mathbf{x}))$ consists in finding $\mathbf{x}^* \in K$ such that:

$$(\mathbf{x} - \mathbf{x}^*)^\top G(\mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in K.$$

We can solve some classes of GNEP by finding a solution for the associated *variational inequality* (VI) problem $\mathrm{VI}(\mathcal{X}, F(\mathbf{x}))$

$$F(\mathbf{x}) = \begin{bmatrix} \nabla_{\mathbf{x}_1} f_1(\mathbf{x}_1, \mathbf{x}_{-1}) \\ \vdots \\ \nabla_{\mathbf{x}_N} f_i(\mathbf{x}_N, \mathbf{x}_{-N}) \end{bmatrix}$$

...thus...

$$(\mathbf{x} - \mathbf{x}^*)^\top F(\mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathcal{X}$$

Monotone operators on VIs

There is a convexity for the map F?

Monotonicity (Scutari et al. 2010)

- If f is convex $\Leftrightarrow$ F is monotone
- If f is strictly convex $\Leftrightarrow$ F is strictly monotone
- If f is strongly convex $\Leftrightarrow$ F is strongly monotone

Solution set (Scutari et al. 2010)

- F monotone: the solution set of VIP$(\mathcal{X}, F(x))$, is closed and convex.
- F strictly monotone: the solution set of VIP$(\mathcal{X}, F(x))$, admits at most one solution.
- F strongly monotone: the solution set of VIP$(\mathcal{X}, F(x))$, admits a unique solution.

Variational inequality problem

The resulting solution, is also a solution of the associated GNEP and is thus called ***variational generalized Nash equilibrium*** (vGNE).

Theorem: Existence (Facchinei et al. 2007)

Let us consider a jointly convex generalized game. A solution of the VIP($\mathcal{X}, F(x)$) **exists**, is **unique** and is a solution of the original GNEP.

Not all the solutions of the GNEP are solutions of the VI but all the solutions of the VI are solutions for the respective GNEP.

The GNEP solutions achieved through the VI are those for which all players' Karush-Kuhn-Tucker (KKT) conditions (optimality conditions) have the same Lagrangian multipliers, i.e., $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}_i$, $\forall i \in \mathcal{N}$.

The equivalence of the Lagrangian multipliers impose to the vGNE having a "fair" behaviour between all the possible GNE.

Game Theory on Nonconvex Sets

Game theory on nonconvex sets

How we can make this problem more difficult?

Game theory on nonconvex sets

- Convexity is required for the existence of a GNE
- Due to the nature of several applications, the coupling feasible set $\mathcal{X}$ may result to be nonconvex.

... really few results available in the literature

The following slides summarize the results of papers:

Scarabaggio, P., Carli, R., and Dotoli, M.
Noncooperative Equilibrium Seeking in Distributed Energy Systems under AC Power Flow Nonlinear Constraints.
IEEE Transactions on Control of Network Systems (con. accepted).

Scarabaggio, P., Carli, R., Grammatico, S., and Dotoli, M.
Clarke's Local Equilibria in Nash Games with Nonconvex Coupling Constraints.
IEEE Transactions on Automatic Control (submitted).

Clarke's local generalized Nash equilibrium

We searched for weaker equilibrium conditions, and we propose a novel concept: Clarke's local generalized Nash equilibrium (CL-GNE).

Our approach relies on the definition of Clarke's tangent cone $\tilde{\mathcal{X}}(\mathbf{x}) := \mathbf{x} + T_{\text{cl}}(\mathcal{X}, \mathbf{x})$ at a point $\mathbf{x}$ for the (nonconvex) set $\mathcal{X}$.

Definition: Clarke's local GNE (Scarabaggio et al. 2022)

A CL-GNE is a collective strategy $\mathbf{x}^* \in \mathcal{X}$ such that for each $i \in \mathcal{N}$:

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq \inf \left\{ f_i(\mathbf{y}, \mathbf{x}_{-i}^*) \mid \mathbf{y} \in \tilde{\mathcal{X}}_i(\mathbf{x}_{-i}^*) \right\}$$

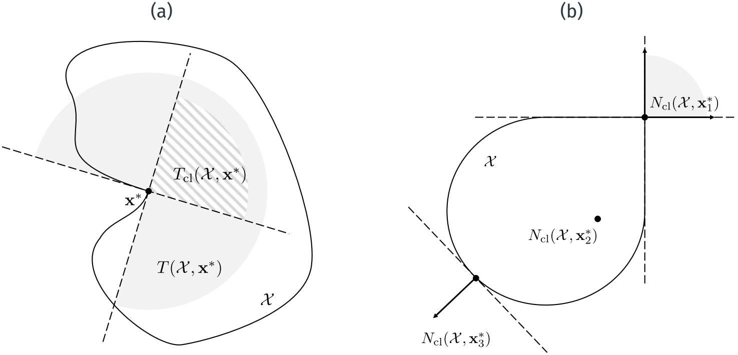
where $\tilde{\mathcal{X}}_i(\mathbf{x}_{-i})$ is the tangent cone of the i-th player which depends also on the other players.

- ▷ If $\mathcal{X}$ is convex, the CL-GNE is equivalent to the GNE
- ▷ Clarke's tangent cone is always convex even if $\mathcal{X}$ is nonconvex

Clarke's tangent cone

Definition: Clarke's tangent cone (Jahn 1996)

Let $\mathcal{X}$ be a nonempty subset of a Euclidean space E. The set of all Clarke's tangent vectors to $\mathcal{X}$ at $\mathbf{x}$ is called *Clarke's tangent cone* of the subset $\mathcal{X}$ at $\mathbf{x}$ and is defined as $T_{cl}(\mathcal{X}, \mathbf{x})$.



Clarke's local generalized Nash equilibrium

Assumption 1: cost functions

For each $i \in \mathcal{N}$ and for every $\mathbf{x}_{-i}$, the function $f_i(\cdot, \mathbf{x}_{-i})$ is convex and continuously differentiable.

Assumption 2: coupling feasible set

For each $m \in \mathcal{M}$ and for every $\mathbf{x}_{-i}$, the function $g_m(\cdot, \mathbf{x}_{-i})$ is continuously differentiable (possibly nonconvex). The coupled feasible set $\mathcal{X}$ is nonempty and compact.

Theorem: Characterization (Scarabaggio et al. 2022)

If $\mathbf{x}^* \in \mathcal{X}$ is a CL-GNE, we have that for each $i \in \mathcal{N}$:

(i) $-\nabla f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \in N_{\text{cl}}(\mathcal{X}_i(\mathbf{x}_{-i}^*), \mathbf{x}_i^*)$;

(ii) there exists a vector $\boldsymbol{\lambda}_i \in \mathbb{R}_{\geq 0}^M$ satisfying the KKT conditions.

We cannot define the variational inequality problem since the tangent cone is different at each point. Thus, we consider a quasi-variational inequality (QVI) associated with the CL-GNEP.

Definition: Quasi-variational inequality (Noor 2006)

Given the tangent cone $\tilde{\mathcal{X}}(\mathbf{x})$ and the mapping F, the *quasi-variational inequality problem* $\text{QVIP}(\tilde{\mathcal{X}}(\mathbf{x}), F(\mathbf{x}))$ consists in finding a vector $\mathbf{x}^* \in \tilde{\mathcal{X}}(\mathbf{x}^*)$, such that:

$$\inf_{\mathbf{y} \in \tilde{\mathcal{X}}(\mathbf{x}^*)} (\mathbf{y} - \mathbf{x}^*)^\top F(\mathbf{x}^*) \geq 0.$$

We define the *variational Clarke local generalized Nash equilibrium* (vCL-GNE) as a solution of the CL-GNEP that satisfies the QVIP.

Variational Clarke's local generalized Nash equilibrium

Similarly to the relation between GNEP and VIP, not all the solutions of the CL-GNEP are solution of the QVIP; viceversa, a solution of the QVIP is solution of the original CL-GNEP.

Theorem: Characterization (Scarabaggio et al. 2022)

(i) Let $\mathbf{x}^*$ be a solution of the CL-GNEP, where the KKT conditions for all players hold with the same Lagrangian multipliers $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i, \forall i \in \mathcal{N}$. Then, $\mathbf{x}^*$ is a solution of the QVI and thus it is a vCL-GNE.

(ii) Viceversa, let $\mathbf{x}^*$ be a solution of the QVI and thus be a vCL-GNE. Then, $\mathbf{x}^*$ is a solution of the CL-GNEP at which the KKT conditions hold with the same Lagrangian multipliers, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_i, \ \forall i \in \mathcal{N}$.

At a vCL-GNE, we have that in a local subset of $\mathcal{X}$ each agent cannot unilaterally maximize their own function while keeping the strategies of the other agents fixed (locally fair equilibrium point).

Existence and local uniqueness

Since the projection onto a nonconvex set is not a nonexpansive operator, classical existence and convergence proof based on projected gradient approaches does not apply here. Thus, we focus on prox-smoth nonconvex sets, to employ the weaker proprieties of the projection operator.

Proposition: Existence (Scarabaggio et al. 2022)

Let Assumptions 1 and 2 hold and let the set $\mathcal{X}$ be r-proximally smooth. Then, the CL-GNEP has at least one vCL-GNE.

Proposition: Local uniqueness (Scarabaggio et al. 2022)

Under Assumption 1 and 2, if the mapping F is strictly monotone, then the strict inequality holds and thus any vCL-GNE $\mathbf{x}^* \in \mathcal{X}$ is unique in its Clarke's tangent cone $\tilde{\mathcal{X}}(\mathbf{x}^*)$.

Convergence of solution algorithms

Convergence to a vCL-GNE demonstrated under (stronly) monotone pseudo-gradient mappings for the classical projected and for a modified version of the Korpelevich's approach.

Assumption 3: stronly monotone pseudo-gradient mappings

The set $\mathcal{X}$ is r-proximally smooth. The mapping F is strongly monotone with constant $\mu > 0$ and Lipschitz continuous with constant $\ell > 0$.

$$\mathsf{x}^{k+1} = \mathrm{proj}_{\mathcal{X}}(\mathsf{x}^k - \gamma F(\mathsf{x}^k))$$

Assumption 4: monotone pseudo-gradient mappings

The set $\mathcal{X}$ is r-proximally smooth. The mapping F is monotone and Lipschitz continuous with constant $\ell > 0$.

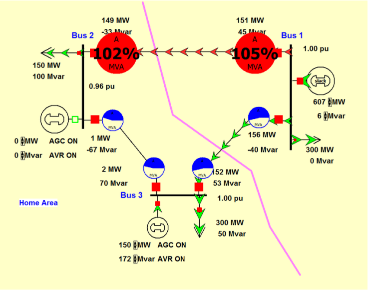
$$\mathsf{x}^{k+1} = \mathrm{proj}_{\tilde{\mathcal{X}}(\mathsf{y}^k)}(\mathsf{x}^k - \gamma F(\mathrm{proj}_{\mathcal{X}}(\mathsf{x}^k - \gamma F(\mathsf{x}^k)))$$

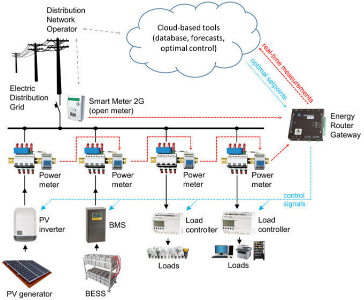
Motivating Examples and Analysis of Case Studies

Nonconvex constraints: AC power flow

$$|V_b| \sum_{r \in \mathcal{B}} |V_r||Y_{br}| \cos(\theta_{br} + \theta_r - \theta_b) = P_b$$

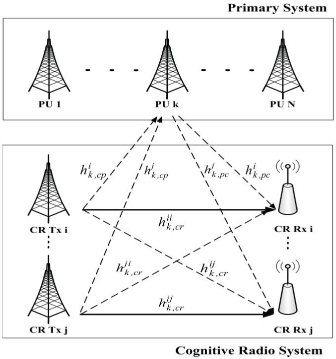
$$-|V_b| \sum_{r \in \mathcal{B}} |V_r||Y_{br}| \sin(\theta_{br} + \theta_r - \theta_b) = Q_b$$

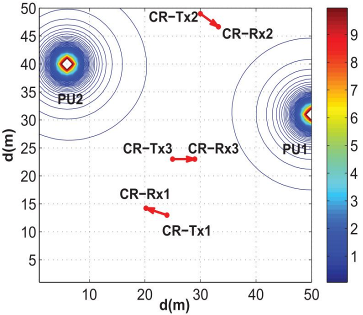




Nonconvex Power Allocation Games in Cognitive Radios

Nonconvex constraints: power allocation in Gaussian frequency-selective interference channels

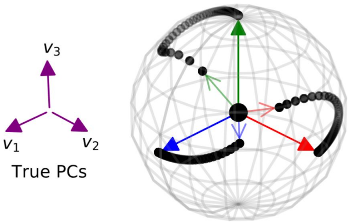




Eigengame: PCA as a Nash equilibrium

Principal component analysis (PCA) as a competitive game in which each approximate eigenvector is controlled by a player whose goal is to maximize their own utility function

parallelizing the computation by transforming the problem in a noncooperative game



Conclusions and Research Outlooks

Conclusions and Research Outlooks

- Definition of the problem ✓
- Optimality of the solution ✓
- Existence with smoothness proprieties ✓
- Uniqueness with smoothness proprieties ✓
- Convergence to a vCL-GNE on monotone operators ✓
- Convergence to a vCL-GNE on strongly monotone operators ✓
- Existence on more general settings ×
- Distributed convergence ×
- And so on...

Scarabaggio, P., Carli, R., Grammatico, S., and Dotoli, M.
Clarke's Local Equilibria in Nash Games with Nonconvex Coupling Constraints.
IEEE Transactions on Automatic Control (submitted).

Scutari, G., Palomar, D. P., Facchinei, F., and Pang, J.-S. (2010).
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IEEE Signal Processing Magazine, 27(3):35–49.

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