Maximizing Convergence Speed for Second Order Consensus in Leaderless Multi-Agent Systems

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Outline

- Consensus Problem Statement
- Consensus Protocol
- Maximizing Convergence Speed
- Numerical Results and Comparison
- Conclusions and Future Research

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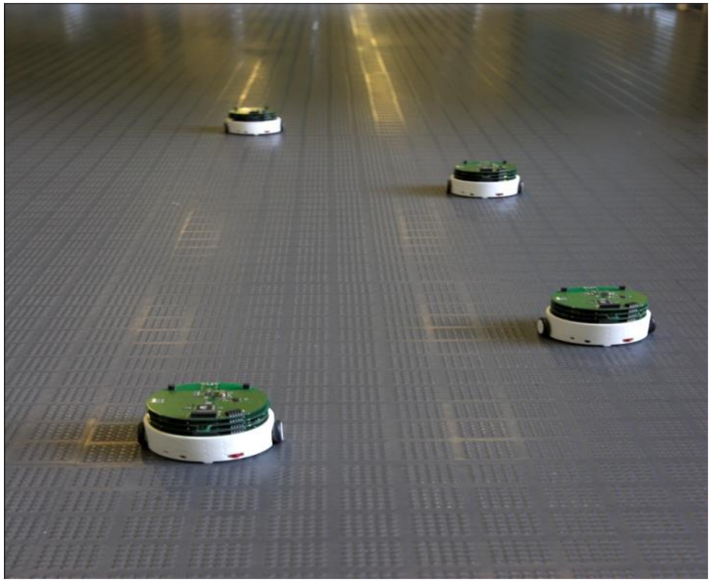
Consensus Problem Statement

- Consensus is the reaching of an agreement on some quantity of interest, by a group of entities.
- Agents try to reach agreement on a common value by exchanging tentative values and combining them.
- Consensus is applied to different fields including cooperative control of unmanned air vehicles, mobile robots, autonomous underwater vehicles, satellites, aircraft, spacecraft, and automated highway systems.



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Consensus Problem Statement

- Agents must reach a **common speed**.
- Agents must form a **uniformly spaced string**
- Agents decide the value of the final common velocity through a consensus protocol, starting from an initial desired value for each agent.

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Consensus Problem Statement

- Both the **leaderless consensus** and the **leader-following** consensus problems have been studied
- It depends on whether or not there is a virtual leader specifying the global information
- The presented consensus protocol **is lederless**
- **Leaderless consensus** scales better and is more fault-tolerant than leader-following consensus.
- **Leader decreases the degree of autonomy** of the network, in many practical missions, the agents need to reach autonomous agreement.

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Consensus Problem Statement

- The interaction topology of a network of agents is represented using a directed graph $G=(V,E)$:
  - $V=\{1,2,..,n\}$ set of nodes
  - $E \subseteq V \times V$ set of edges.

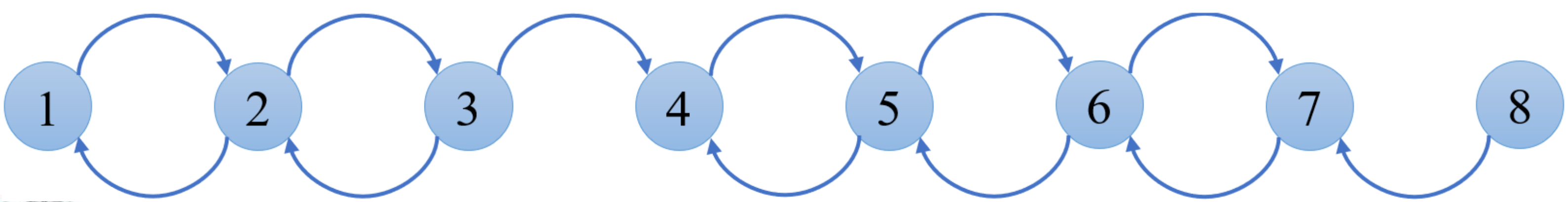
- $\boldsymbol{A}=[a_{ij}]$ is the adjacency matrix
- The Laplacian matrix $\boldsymbol{L} = [l_{i,j}]$ is an $n \times n$ matrix derived from $\boldsymbol{A}$:

$$l_{i,i} = -\sum_j a_{ij} \text{ and } l_{i,j} = -a_{ij} \text{ if } i \neq j.$$

$(i, j) \in E$ if agent i can receive information from agent j

i is the child

j is the parent



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Consensus Problem Statement

$(i, j) \in E$ if agent i can receive information from agent j

i is the child

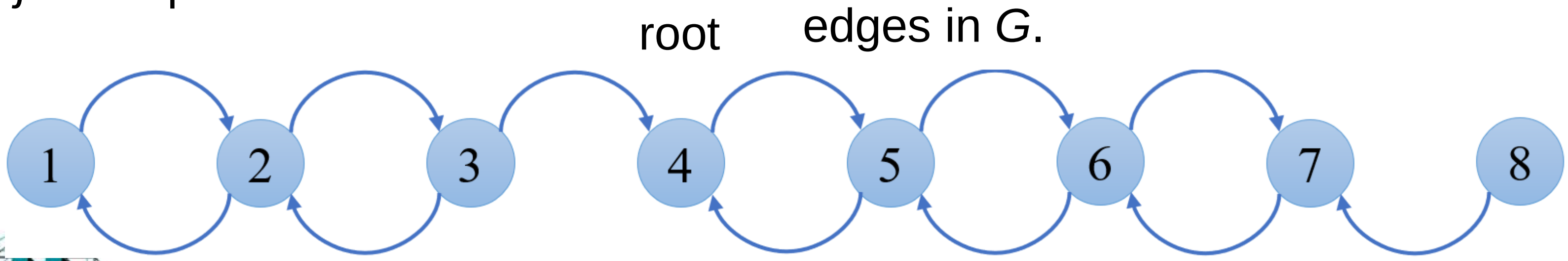
j is the parent

directed path from j to i*:* a sequence of edges (i, i_1), (i_1, i_2), ..., (i_l, j) with distinct vertices i_k, $k = 1, 2, ..., l$.

root r: a vertex r such that for each vertex i different from r, there is a directed path from r to i.

directed tree: a digraph in which there is exactly one root and each vertex except for the root has exactly one parent.

directed spanning tree: a directed tree, which consists of all the vertices and some edges in G.



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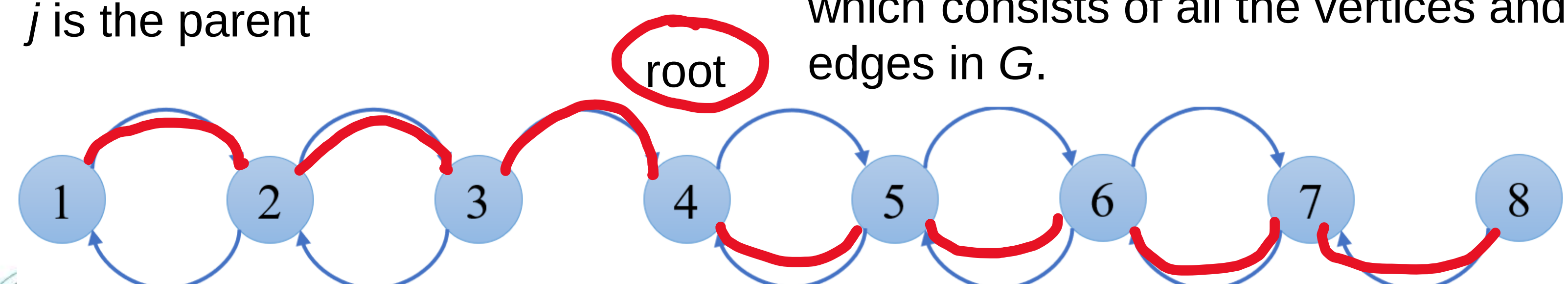

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Consensus Problem Statement

Consider vehicles with double-integrator dynamics:

x_i= position of agent *i*

v_i= velocity of agent *i*

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{O}_n & \boldsymbol{I}_n \\ \boldsymbol{O}_n & \boldsymbol{O}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{u} \end{bmatrix}.$$

Multi-agent system control problem:

i) each agent must reach and steadily keep a common reference velocity v

ii) all the agents must be spaced with uniform interspace gap *d*.

The common velocity is unknown to the agents. Agent i starts from an initial value $y_i(0)$ for i = 1, ..., n of the reference velocity and by the consensus protocol the agents reach a common value of y_i for i=1,...,n.

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The starting point:
the consensus protocol proposed by Ren (Trans. on AC, 2008)
successively other autors provide new contributions considering delays in the communication or actuator saturation in the control.
(r_i is the position).

$$u_i = -\sum_{j=1}^{n} a_{ij}[(r_i - r_j) + \gamma(v_i - v_j)], \quad i \in \mathcal{I}_n$$

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Consensus Protocol

The consensus algorithm:

$$u_i = -\sum_{j\in\mathcal{N}(i)} a_{ij}[(x_i - x_j) - \bar{d}(i-j) - \gamma(v_i - v_j)] -$$
$$-\kappa\,(v_i - y_i)\ \text{ with } \gamma, \kappa \in \mathbb{R}^+.$$

- by the first terms each agent communicates the actual distance from its neighbour and the objective inter-space to be imposed between two nearby agents;
- by the second terms the agents communicate the actual difference between the velocities of its neighbours and the reference velocity

$$\dot{y}_i = -\eta \sum_{j\in\mathcal{N}(i)} a_{ij}\,(y_i - y_j),\ \text{ with } \eta \in \mathbb{R}^+.$$

Dynamics of the reference velocity

Consensus Protocol

The consensus algorithm:

$$u_i = -\sum_{j\in\mathcal{N}(i)} a_{ij}[(x_i - x_j) - \bar{d}(i-j) - \gamma(v_i - v_j)] - \\ -\kappa\,(v_i - y_i) \;\; \text{with } \gamma, \kappa \in \mathbb{R}^+.$$

Dynamics of the system:

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{O}_n & \boldsymbol{I}_n & \boldsymbol{O}_n \\ -\boldsymbol{\mathcal{L}} & -(\gamma\boldsymbol{\mathcal{L}} + \kappa\boldsymbol{I}_n) & \kappa\boldsymbol{I}_n \\ \boldsymbol{O}_n & \boldsymbol{O}_n & -\eta\boldsymbol{\mathcal{L}} \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{v} \\ \boldsymbol{y} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_n \\ \boldsymbol{\mathcal{L}}\boldsymbol{H}^+\bar{\boldsymbol{d}} \\ \boldsymbol{0}_n \end{bmatrix} \tag{1}$$

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Consensus Protocol

The consensus algorithm:

$$u_i = -\sum_{j \in \mathcal{N}(i)} a_{ij}[(x_i - x_j) - \bar{d}(i-j) - \gamma(v_i - v_j)] - \\ -\kappa\,(v_i - y_i) \;\text{ with } \gamma, \kappa \in \mathbb{R}^+.$$

Defining the following new variables:

$\boldsymbol{p}$=$\boldsymbol{Hx}$-$\boldsymbol{d}$ $\boldsymbol{q}$=$\boldsymbol{v}$-$\boldsymbol{y}$ $\mathbf{r}$=$\boldsymbol{Hy}$ with

$$\boldsymbol{H} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots. & 1 & -1 \end{bmatrix}.$$

The dynamics of the system is described by:

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{r}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{O}_{n-1} & \boldsymbol{H} & \boldsymbol{I}_{n-1} \\ -\boldsymbol{\mathcal{L}}\boldsymbol{H}^+ & -(\gamma\boldsymbol{\mathcal{L}} + \kappa\boldsymbol{I}_n) & (\eta - \gamma)\,\boldsymbol{\mathcal{L}}\boldsymbol{H}^+ \\ \boldsymbol{O}_{n-1} & \boldsymbol{O}_{n-1,n} & -\eta\boldsymbol{H}\boldsymbol{\mathcal{L}}\boldsymbol{H}^+ \end{bmatrix}}_{\boldsymbol{F}} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix}.$$

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Consensus Protocol

The multi-agent system control problem is equivalent to making system the following system **asymptotically stable.**

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{r}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{O}_{n-1} & \boldsymbol{H} & \boldsymbol{I}_{n-1} \\ -\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} & -\left(\gamma\boldsymbol{\mathcal{L}} + \kappa\boldsymbol{I}_{n}\right) & \left(\eta-\gamma\right)\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} \\ \boldsymbol{O}_{n-1} & \boldsymbol{O}_{n-1,n} & -\eta\boldsymbol{H}\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} \end{bmatrix}}_{\boldsymbol{F}} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix}.$$

To be proved: the conditions that γ and κ must satisfy in order to ensure that the system described by ***F*** is asymptotically stable

the consensus protocol successfully solves the multi-agent system control problem.

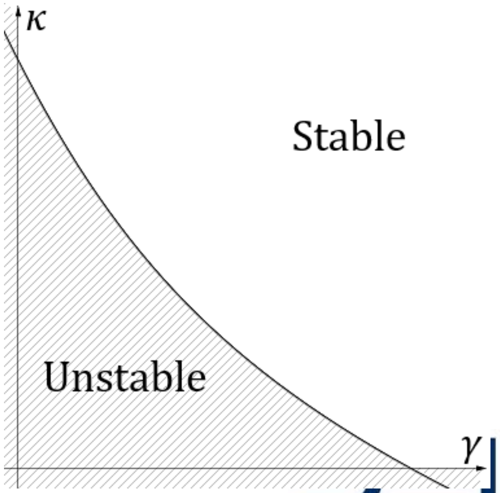
Consensus Protocol

***Theorem 1**:* Consider a set of agents that communicate in a network topology described by a digraph *G* that has a directed spanning tree. The dynamics of the multi-agent system is asymptotically stable if and only if it holds:

$$\alpha_i \left(\alpha_i^2 + \beta_i^2\right) \gamma^2 + \left(2\alpha_i^2 + \beta_i^2\right) \gamma\kappa + \alpha_i \kappa^2 - \beta_i^2 > 0$$

for i = 1,...,n−1, with $\alpha_i = Re[\mu_i]$ and $\beta_i = Im[\mu_i]$ where μ_i are the eigenvalues of the Laplacian matrix of digraph *G*.
(*Re*[*c*] and *Im*[*c*]) denote the real and imaginary part of complex number *c*, respectively).

The points (κ, γ) to obtain stabilty must lie beyond the ”critical hyperbolae”.



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Maximizing Convergence Speed

The control problem objective

maximizing the consensus protocol convergence speed and avoiding large oscillations

$$\begin{bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{r}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{O}_{n-1} & \boldsymbol{H} & \boldsymbol{I}_{n-1} \\ -\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} & -\left(\gamma\boldsymbol{\mathcal{L}} + \kappa\boldsymbol{I}_n\right) & \left(\eta - \gamma\right)\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} \\ \boldsymbol{O}_{n-1} & \boldsymbol{O}_{n-1,n} & -\eta\boldsymbol{H}\boldsymbol{\mathcal{L}}\boldsymbol{H}^{+} \end{bmatrix}}_{\boldsymbol{F}} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix}.$$

a) selecting a real dominant eigenvalue of matrix ***F*** and maximizing the its absolute value

b) allocating the not dominant eigenvalues as far away as possible from the imaginary axis

choosing suitable values of parameters γ and κ

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Maximizing Convergence Speed

Proposition: Consider a set of agents that communicate in a network topology described by a digraph *G* that has all the strongly connected components symmetric and a directed spanning tree.

Let $\mu_0 = 0$ and $\mu_i \in R+$ for i = 1,...,n be the eigenvalues of the Laplacian matrix of *G* arranged in increasing order with *i*.

The eigenvalues that solve the control problem are obtained by the following values of the parameters:

$$\bar{\gamma} = \frac{2\sqrt{\mu_1}}{\mu_1 + \mu_2}, \quad \bar{\kappa} = \frac{2\mu_2\sqrt{\mu_1}}{\mu_1 + \mu_2}, \quad \bar{\eta} \gg \frac{1}{\sqrt{\mu_1}}.$$

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Numerical Results and Comparison

n=8 agents

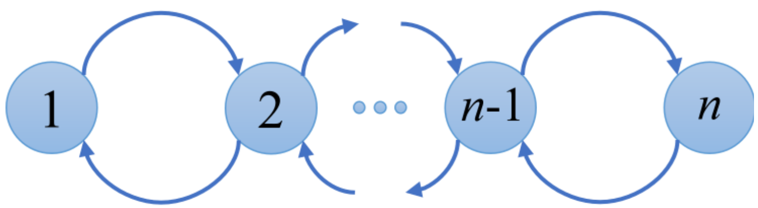
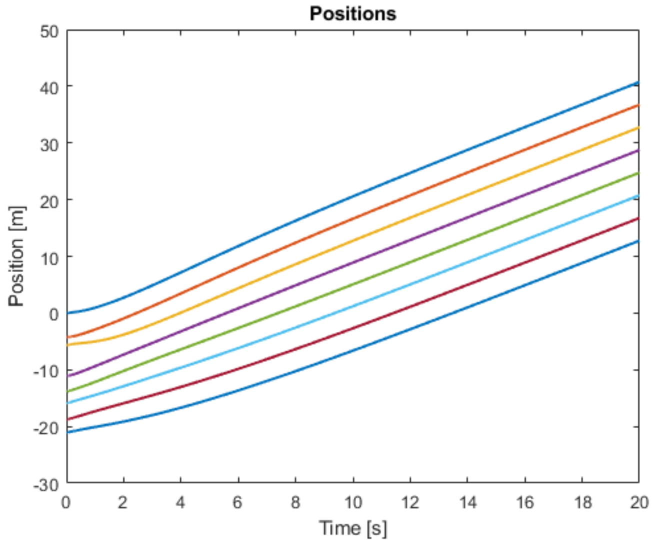
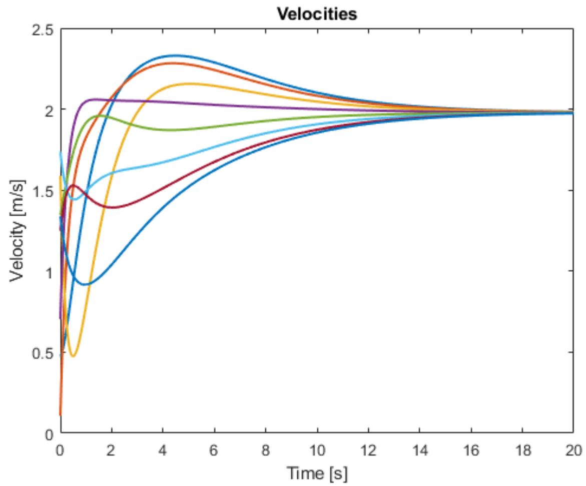


TABLE I
INITIAL CONDITIONS OF THE TESTED SCENARIO AND PARAMETERS

Parameters	Values
$\bar{\gamma}$	1.0574
$\bar{\kappa}$	0.6194
$\bar{\eta}$	25.62
$\bar{d}$	4
$\boldsymbol{d}(0)$	$[4.26, 1.44, 5.40, 2.73, 2.11, 2.85, 2.32]^T$
$\boldsymbol{v}(0)$	$[0.47, 0.10, 1.58, 0.70, 1.35, 1.74, 1.25, 1.34]^T$
$\boldsymbol{y}(0)$	$[1.94, 1.62, 1.99, 2.35, 2.37, 1.77, 1.71, 2.07]^T$



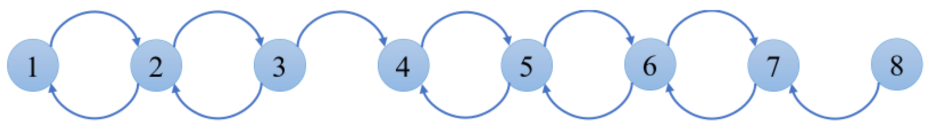
Positions over time.



Velocities over time.

Numerical Results and Comparison

n=8 agents



G_{P*}

G_{P*} has a directed spanning tree and its strongly connected components are symmetric.

By Corollary 1 and Proposition 3 the eigenvalues of *L* are real.

d	4
$\boldsymbol{d}(0)$	$[4.26, 1.44, 5.40, 2.73, 2.11, 2.85, 2.32]^T$
$\boldsymbol{v}(0)$	$[0.47, 0.10, 1.58, 0.70, 1.35, 1.74, 1.25, 1.34]^T$
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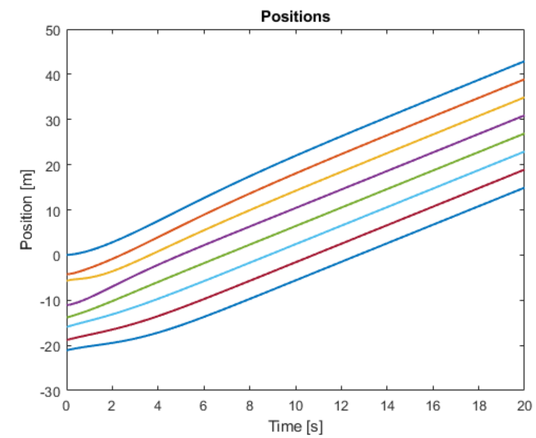
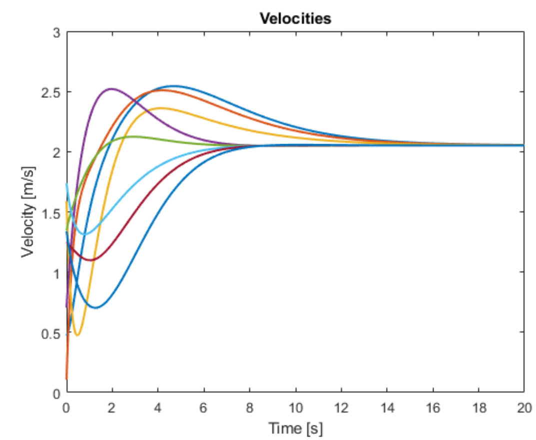


Positions over time for network topology $\mathcal{G}_P^*$.



Velocities over time for network topology$\mathcal{G}_P^*$.

$\bar{\gamma} = 1.1355,\ \bar{\kappa} = 0.6652$ and $\bar{\eta} = \frac{10}{\sqrt{\mu_1}}$ ⟶ The new parameters

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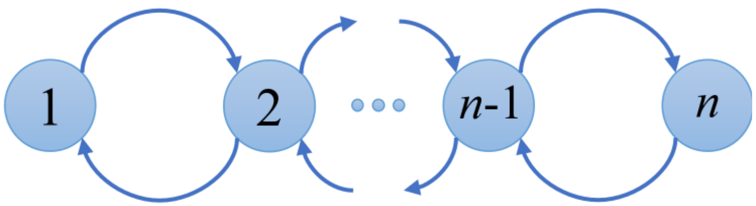
Numerical Results and Comparison

Comparison with a similar method presented in literature:
the two protocols are applied to the graph topology characterized by Laplacian matrices with real non negative eigenvalues μ_i for i = 0, ..., (n-1):



$$u_i = -\sum_{j\in\mathcal{N}(i)} (x_i - x_j) - \gamma_1 \sum_{j\in\mathcal{N}(i)} (v_i - v_j).$$

$$\sqrt{\frac{\mu_1\mu_{n-1}}{2\mu_{n-1} - \mu_1}} = \sqrt{\mu_1}\sqrt{\frac{\mu_{n-1}}{\mu_{n-1} + (\mu_{n-1} - \mu_1)}} < \sqrt{\mu_1}.$$

By introducing the second parameter , the proposed protocol can reach a greater convergence speed than the protocol using only parameter γ.

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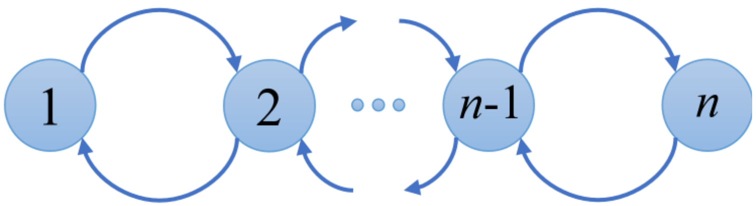
Numerical Results and Comparison



$$u_i = -\sum_{j\in\mathcal{N}(i)} (x_i - x_j) - \gamma_1 \sum_{j\in\mathcal{N}(i)} (v_i - v_j).$$

INITIAL CONDITIONS OF TESTED SCENARIOS FOR THE COMPARISON.

Parameters	Values
$d_i(0)$	$\sim \mathcal{U}(1, 10)$ m
$v_i(0)$	$\sim \mathcal{U}(0,\ 2.5)$ m/s
$y_i(0)$	$\sim \mathcal{U}(1.5, 2.5)$ m/s

$$V(t) = \|\boldsymbol{v}(t) - \frac{1}{n}\mathbf{1}\mathbf{1}^T\boldsymbol{v}(0)\|,$$

$$V(t) \leq 0.005V(0) \quad \forall t \geq t_{0.5\%}.$$

$t_{0.5\%} = 27.05 \quad s$

$t_{0.5\%} = 29.82 \quad s.$

The proposed protocol

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Conclusions and future research

- The consensus protocol can be applied by a multi- agent system in order to reach a **common velocity with desired spacing**.
- The **leaderless agents** are able to reach a consensus about the **reference velocity** by starting from an initial desired value for each agent.
- We prove the conditions that guarantee the consensus control rules allow the agents stably to achieve the decided inter-agent distance and the common velocity.
- The **optimal eigenvalues allocation** is obtained in a closed form of the control parameter values for a class of digraphs having a directed spanning tree and modelling the communication network topology.
- **Advantage of the method**: 1) a leader is not required; 2) by the optimized protocol parameters the fastest convergence speed avoiding oscillations is guaranteed.

Conclusions and future research

- Assessment of the protocol in presence of constraints on agent velocities and accelerations
- Investigation about the impact on the stability and convergence of the delays of communication
- Determine the suitable conditions to guarantee correct behaviour and good performance of the protocol.

Ref.

M.P. Fanti, G. Difilippo, A.M. Mangini, «Maximizing Convergence Speed for Second Order Consensus in Leaderless Multi-Agent Systems», IEEE/CAA Journal of Automatica Sinica

Thank for your attention!

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